Interest and the Number "e"

How does compound interest and simple interest differ?

- **Simple Interest**: interest is only paid on the original amount each year.
- **Compound Interest**: Interest is paid on the original amount, and then this new amount becomes the new principle. This is then reinvested at the same rate. The principle earns annual interest, but also the interest earns interest as well.

Example 1: You have a choice of depositing $500.00 into a bank account that pays 2% simple interest a year or 2% compound interest a year. Your money will be in this account for 5 years. What is the better deal?

Simple Interest: I=prt

$$I = prt$$
$$I = 500 \cdot \frac{2}{100} \cdot 5$$
$$I = 5 \cdot 100 \cdot \frac{2}{100} \cdot 5$$
$$I = 5 \cdot 2 \cdot 5$$
$$I = 50$$

$\therefore$ after 5 years you will have $550

Compound Interest:

Year 1:
$$= 500 + 500(0.02)$$
$$= 500(1 + 0.02)$$

$$\frac{2}{100} = 0.02$$

Year 2:
$$= 500(1+0.02) + 500(1+0.02)(0.02)$$
$$= 500(1+0.02)(1+0.02)$$
$$= 500(1+0.02)^2$$

Year 3:
$$= 500(1+0.02)^2 + 500(1+0.02)^2(0.02)$$
$$= 500(1+0.02)^2(1+0.02)$$
$$= 500(1+0.02)^3$$

Year 4:
$$500(1+0.02)^4$$

Year 5:
$$= 500(1+0.02)^5$$
$$= 500(1.02)^5$$
$$= 552.04$$

$\therefore$ after 5 years you would have $552.04

The better deal would be the account with compound interest.

In general, to derive the compound interest formula (a formula for virtually all financial calculations), let P = principle and r = percentage rate. The subscript on "P" refers to the year.

$$
\begin{aligned}
P_1 &= P + P \cdot r \\
&= P(1+r) \\
P_2 &= P(1+r) + P(1+r) \cdot r \\
&= P(1+r)(1+r) \\
&= P(1+r)^2 \\
P_3 &= P(1+r)^2 + P(1+r)^2 \cdot r \\
&= P(1+r)^2(1+r) \\
&= P(1+r)^3 \\
&\ \ . \\
&\ \ . \\
&\ \ . \\
P_t &= P(1+r)^t
\end{aligned}
$$

t = the number of years
r = percentage rate
P= principle

Many loans/accounts get compounded more than once a year. It could be quarterly (4 times a year), monthly (12 times a year), daily (365 times a year), etc. For each "conversion period" (time between each interest payment) the lender uses the annual interest rate divided by the number of times a year it is compounded. Let this number be "n". Since t = the number of years, then there are $n \cdot t$ conversion periods. This leads us to…

the **Compound Interest Formula**: $A = P(1+\frac{r}{n})^{nt}$, where A is the final amount after t years.

Example 2: The Trust Fund:

A deposit of $5000.00 from Uncle Money Bags was made in a trust fund that pays 7% interest compounded quarterly. The balance will be given to you 30 years after you graduate from college. How much will you receive when the trust fund is finally given to you? How much more would you get if the account were compounded bi-monthly?

Quarterly	Bi-monthly
$P = 5000$ $r = 0.07$ $n = 4$ $t = 30$ $A = P(1+\frac{r}{n})^{nt}$ $A = 5000(1+\frac{0.07}{4})^{4\cdot 30}$ $A = 5000(1.0175)^{120}$ $A = 40095.92$ $\therefore$ The amount you would recieve would be \$40,095.92	$P = 5000$ $r = 0.07$ $n = 24$ $t = 30$ $A = P(1+\frac{r}{n})^{nt}$ $A = 5000(1+\frac{0.07}{24})^{24\cdot 30}$ $A = 5000(1.002916667)^{720}$ $A = 40706.25$ $\therefore$ You would recieve \$610.33 more if compounded bi-monthly

This raises a question. Does money continue to grow if the compounded event happens more and more frequently? Does it continue to grow if you compound the money every minute, every second, every millisecond? Is there a limit to how much it will grow? (*Take a vote at this time*). Let's look at a special case where P = \$1, r = 100%, t = 1 year and what will happen if the number of compounding events becomes increasingly large.

$A = P(1+\frac{r}{n})^{nt}$, where P = \$1, r = 100%, t = 1 year and n = number of times a year compounded	
n = 1	$1\left(1+\frac{1}{1}\right)^{1} = 2$
n = 2	$1\left(1+\frac{1}{2}\right)^{2} = 2.25$
n = 10	$1\left(1+\frac{1}{10}\right)^{10} = 2.59374246$
n = 100	$1\left(1+\frac{1}{100}\right)^{100} = 2.704813829$
n = 1,000	$1\left(1+\frac{1}{1000}\right)^{1000} = 2.716923932$
n = 10,000	$1\left(1+\frac{1}{10{,}000}\right)^{10{,}000} = 2.718145927$
n = 100,000	$1\left(1+\frac{1}{100{,}000}\right)^{100{,}000} = 2.71826823$
n = 1,000,000	$1\left(1+\frac{1}{1{,}000{,}000}\right)^{1{,}000{,}000} = 2.718280469$
n = 100,000,000	$1\left(1+\frac{1}{100{,}000{,}000}\right)^{100{,}000{,}000} = 2.718281815$
$n = \infty$	$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = 2.718281828\ldots$

It was in the 1600 that a mathematician, a merchant or a moneylender noticed the way money grows in this fashion. This "number" received a lot of attention and eventually was given a name.
The mathematician, Leonhard Euler (pronounced "Oiler"), gave it the symbol "*e*" and it is sometimes referred to as the Euler Number. The earliest appearance of this symbol was in 1736 in the book *Mechanica* written by Euler. There is controversy as to why he picked "*e*". Some say it was for the word "exponent", some say it was because it was the next letter in the alphabet that wasn't being used (*a, b, c* and *d* were being used frequently) and most unlikely for his last name (he was a very modest man). Euler was to first to prove that "*e*" was an irrational number in 1737. [1]

1. Maor, E (1998) *The Story of a Number*. Princeton University Press, Princeton New Jersey.

Definition of the number *e*: $e \approx 2.718281828459\ldots$ is an irrational number that occurs in advanced mathematics, economics, statistics, probability and other situations involving growth.

When we compound continuously our number converges to the number *e*. What will this do to our formula?

Let ***e*** be defined as $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = 2.718281828\ldots$

Let $u = \frac{n}{r}, \quad \therefore n = u \cdot r$

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Substitute $n = ur$

$$A = P\left(1+\frac{r}{ur}\right)^{urt}$$

$$A = P\left(1+\frac{\cancel{r}}{u\cancel{r}}\right)^{urt}$$

Oh, look, an equivalent form of 1!

$$A = P\left(1+\frac{1}{u}\right)^{urt}$$

$$A = P\left(\left(1+\frac{1}{u}\right)^{u}\right)^{rt}$$

Using laws of exponents.

$$A = P\left(e\right)^{rt}$$

Substitute in *e* for the definition.

So, we now have a new formula that we can use to find continuous compounding.

Continuous Compound Interest formula : $A = P\left(e\right)^{rt}$ where A is the final amount after t years.

Example 3: The Trust Fund (take two)

The deposit of $5000.00 from Uncle Money Bags was made in a trust fund that pays 7% interest compounded daily. The balance will be given to you 30 years after you graduate from college. How much will you receive when the trust fund is given to you? How much more would you get then if it were compounded continuously?

Daily (you try)	**Continuously (we try)**
$P = 5000$ $r = 0.07$ $n = 365$ $t = 30$ $A = P(1+\frac{r}{n})^{nt}$ $A = 5000(1+\frac{0.07}{365})^{365\cdot 30}$ $A = 5000(1.0001917808)^{10950}$ $A = 40822.62$ $\therefore$ The amount you would recieve would be \$40,822.62	$P = 5000$ $r = 0.07$ $t = 30$ $A = Pe^{rt}$ $A = 5000e^{0.07\cdot 30}$ $A = 5000e^{2.1}$ $A = 40830.85$ $\therefore$ The amount you would recieve would be \$40,830.85

Warm-Up:

CST/CAHSEE:	Review:
CAHSEE released test question: **Sally puts $200.00 in a bank account. Each year the account earns 8% simple interest. How much interest will be earned in three years?** **A** $16.00 **B** $24.00 **C** $48.00 **D** $160.00	Factor: a. $2x(x+1)+(x+1)$ b. $50(1+x)+50(1+x)x^2$

Current:	Other:
Let $w=12,\ r=3,\ n=4,$ find the value of $w\left(1+\frac{1}{n}\right)^r$	Use the laws of exponents to simplify: a. $\left((x+2)^w\right)^{yt}$ b. $(x+1)^3(x+1)^2$

Today's Objective/Standards: Alg 2 12.0: Students know the laws of fractional exponents, understand exponential functions, and use these functions in problems involving exponential growth and decay.

Warm-Up: (answer key)

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Current:	Other:
Let $w=12,\ r=3,\ n=4,$ find the value of $w\left(1+\frac{1}{n}\right)^r$ $=12\left(1+\frac{1}{4}\right)^3$ $=12\left(\frac{5}{4}\right)^3$ $=12\left(\frac{125}{64}\right)$ $=23.4375$	Use the laws of exponents to simplify: a. $\left((x+2)^w\right)^{yt}$ $=\left(x+2\right)^{wyt}$ b. $\left(x+1\right)^3\left(x+1\right)^2$ $=\left(x+1\right)^5$

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